On the probability of pure Nash equilibria

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Outline

- pure strategy Nash equilibria (*PNE*) need not exist in a normal form game
- *PNE* easier to interpret than a mixed equilibrium
- how likely it is to have a *PNE* in a game?
- here: probability of a *PNE* in 'large' $K \times K$ games
- probability of a *PNE* depends on how fast the number of possible utilities grows *w.r.t* K:
- slow growth $\longrightarrow$ high prob; fast growth $\longrightarrow$ low prob
- limit games: countably infinite action sets
- search for limit distributions of number of *PNE*

Related Literature

- take a $K \times K$ bimatrix game,
- to each action profile (a, b) choose payoffs (x, y) randomly
  - x and y *i.i.d.*; *continuous* distribution; convex support
- Goldberg *et al.* (1968): probability of a *PNE* is

$$1 - e^{-1}$$

  as K goes to infinity
- Dresher (1970): holds for n -person games
- Stanford (1995): number of *PNE* has Poisson distribution with mean 1

- Powers (1990): payoffs from $\{1/K^2, 2/K^2, \ldots, 1\}$, with equal probabilities $1/K^2$
- number of *PNE* still has Poisson dist'n with mean 1 and prob of a *PNE* is $1 - e^{-1}$
- Powers: since pure equilibria are searched for, only ordinal preferences matter
- K^2 payoffs suffices to represent all ordinal preferences over K^2 strategy profiles
- but if pure equilibria are searched, then it's enough to have two payoffs

- let U be a payoff matrix, and let U^B be another matrix with payoffs 0 and 1:
  player 1 gets payoff 1 from action profile (a, b) iff a is a best reply against b in game U

	a	b	c
a	$1, 3$	$3, 2$	$2, 1$
b	$2, 3$	$1, 2$	$1, 3$
c	$1, 2$	$3, 3$	$3, 1$

U

	a	b	c
a	$0, 1$	$1, 0$	$0, 0$
b	$1, 1$	$0, 0$	$0, 1$
c	$0, 0$	$1, 1$	$1, 0$

U^B

- U and U^B have the same *PNE* (b, a) and (c, b) but different limit results

- further: the limit game in all previous papers has action sets $\mathbb{N}$ and payoffs *i.i.d* from uniform dist'n over $[0, 1]$
- this game has no *PNE* with probability 1
- although has (infinitely many) ε -equilibria for every $\varepsilon > 0$ with probability 1
- what is the correct way to go to limit?

Best Reply Equivalence

- games U and V are *BR* -equivalent if they have the same pure best replies, or $U^B = V^B$
- for 2×2 games U, there are $9^2 = 81$ different games U^B
- only two of them have *no PNE*:
- U^B:

	c	d
c	$0,1$	$1,0$
d	$1,0$	$0,1$

  and its ”transpose”
- probability of a *PNE* is $79/81$ in this example

- fix K and choose payoffs from $\{0, 1\}$ independently (50-50) to each player and action profile
- probability of payoff vector $(1, 1)$ goes to 1 as K grows, because the prob of its complement is $(3/4)^{KK}$

$\Longrightarrow$

Proposition 3.1.

The probability that a randomly chosen U with $0 - 1$ -payoffs has a PNE with payoffs $(1, 1)$ goes to 1 as K goes to infinity.

- how to extend this to BR -equivalence classes U^B?

- U^B -matrix looks like a game with $0-1$ payoffs
- except that a player cannot get 0 from *all* choices, given opponent's choice
- as $K \longrightarrow \infty$ there is no difference between U^B -matrices and games with $0-1$ payoffs

  $\Longrightarrow$

Proposition 3.2.

The probability that a randomly chosen U^B has a PNE goes to 1 as K goes to infinity.

Linear Growth

- suppose payoffs are drawn from $\{1/L, 2/L, \ldots, 1\}$ and $L = rK$, for constant $r > 0$
- probability of payoffs $(1, 1)$ in a $K \times K$ game is

$$1 - \left(1 - \frac{1}{r^2 K^2}\right)^{K^2}$$

- converges to $1 - e^{-1/r^2}$ as $K \longrightarrow \infty$
- the probability of a *PNE at least* this number

- the probability of a *PNE* goes to 1 as $r \longrightarrow 0$
- when $r = 0$, number of payoffs is fixed and finite and expected number of *PNE* is infinite
- we don't know the limit distribution for the number of *PNE* when $0 < r < \infty$
- but the limit distribution for *PNE* with payoffs $(1, 1)$ is Poisson with mean $1/r^2$

Nonlinear Growth

- suppose number of payoffs is $L = K^{\alpha}, \ \alpha > 0$
- Powers (1990): if $\alpha > 1$ then the number of *PNE* is Poisson distributed with mean 1, as $K \longrightarrow \infty$
- if $\alpha < 1$ then the probability of a payoff vector $(1, 1)$ goes to 1, as $K \longrightarrow \infty$
- and the expected number of *PNE* goes to infinity