Randomly Matched Bargaining for One Unique Good

Andreas Nohn

Public Choice Research Centre
Turku, Finland

Institute of SocioEconomics
University of Hamburg, Germany

nohn@econ.uni-hamburg.de

Abstract

We investigate noncooperative n-player bargaining for one unique good. In every round, a proposer and a coalition containing him are chosen at random. The proposer suggests a division of the unique good which is implemented if all members of the coalition approve. Otherwise, future payoffs are discounted and the game starts all over.

As an extension of similar work, both the probability distribution for assigning the role of the proposer as well as each proposer's probability distribution over the possible responding coalitions may take arbitrary values. In addition, it is allowed for different time preferences between players.

We show existence and uniqueness of stationary subgame perfect equilibria (SSPE). A variety of basic properties as well as monotonicities of the corresponding payoffs are derived. Finally, we consider the limit of payoffs for equal discount factors converging to 1 the players' bargaining power.

1. Introduction

In many social and economic situations, coalitions from a set of players may jointly pursue an activity which – in spite of the corresponding costs – generates a surplus. In case that no exogenous mechanism exists to determine the distribution of that surplus, players have to bargain for their piece of the pie. Possible examples include the sale of a good, joint ventures, or political policies.

In this paper, we consider n players bargaining noncooperatively for the surplus generated by one unique activity. That is, no more than one coalition can pursue the activity and the game ends after one coalition has done so successfully. We further assume the generated surplus to be 1 for each possible coalition, and refer to this as the unique good. As the main feature of the game, we assume that communication between players is random and exogenously restricted. In each round, one player is randomly assigned the role of the proposer. Then, conditional on this player being the proposer, a coalition is chosen at random which has the opportunity to set up the unique activity. The proposer suggests a division of the surplus of 1 which is, in some arbitrary order, to be rejected or accepted by each member of the coalition. If at least one member declines, the game proceeds to the next round and all future payoffs are individually discounted. If all members agree, the surplus is divided accordingly, players outside of the coalition receive nothing, and the game ends. In order to predict the players' behavior, we restrict attention to stationary subgame perfect equilibria (SSPE) for which existence and uniqueness is shown. Subsequently, properties of the unique SSPE payoffs are investigated, among them smoothness in all parameters, efficiency, null players, and monotonicities for each group of parameters. Some supposedly intuitive monotonicties do, however, not show in all cases. Finally, payoffs of infinitely patient players are found to be interpretable as bargaining power. While some properties of the payoffs are not valid for bargaining power, an additional property concerning veto players usually having all power is found.

The main influence for this work comes from bargaining models in the field of social networks. In these, communication possibilities between individuals are characterized by an unweighted and undirected graph. In a cooperational setting, Myerson (1977) was the first to explicitly model restricted communication. For TU games where coalitions are limited to those connected by an undirected and unweighted graph, he derives and axiomatizes a coalitional value closely related to the Shapley value. Other coalitional values have been investigated by Owen (1986) and Rosenthal (1988), among others. So far, restricted communication has received much less attention in noncooperative game theory. In Berg (1999), any connected pair of players has equal probability to engage in bargaining for one of multiple goods, and either player has equal probability for being the proposer. Whenever two players agree on a division, they leave the game and bargaining continues in the remaining subgraph. Calvó-Armengol (2001) extends Rubinstein (1982) to n-player bilateral bargaining for one unique good. In the first round, each player has equal probability to be the proposer and each of his neighbors has equal probability to be the respondent. In case the respondent agrees on the division, the game ends. If he rejects the offer, the game starts all over with the respondent being the new proposer.

In these models, bilateral communication possibilities are induced by undirected and unweighted graphs which naturally impose symmetry in some sense or other. In contrast, this paper's main focus rests arbitrary probability distributions. In addition, the dynamics of matching allow for coalitional bargaining with no restrictions whatsoever to the possible coalitions. Thus, communication structures in this work resemble weighted and directed hypergraphs.

A second field in the literature having apparent similarities to this work is that of noncooperative coalitional bargaining models. While an underlying TU game determines the value of each coalition, the game's course is usually similar to ours but that proposers are allowed to choose freely which coalition to propose to. Most classic, the model of Baron and Ferejohn (1989) assumes a supermajority voting rule, equal probability for each player to be the proposer, and equal discount factors in order to show existence of a SSPE. Uniqueness and further properties of the SSPE payoffs are shown by Eraslan (2002) also for arbitrary proposer probabilities and discount factors. Okada (1996) shows existence of an SSPE with non-delayed agreement in the grand coalition for TU games with increasing per-capita values of coalitions.

While the free choice of coalitions of course seems a desirable option for a bargaining game, there is not yet any model offering statements of existence and uniqueness of equilibria for general dynamics of matching.

The remainder of the paper is organized as follows. Section 2 presents the randomly matched bargaining game by stating the relevant parameters as well as the bargaining procedure. In section 3, stationary subgame perfect equilibria are characterized by the equilibrium payoffs and their existence and uniqueness are shown. Basic properties and monotonicities are presented in section 4. Section 5 considers bargaining power as the payoffs of infinitely patient players and lists corresponding properties. We conclude in section 6.

2. The randomly matched bargaining game

The randomly matched bargaining game we consider is noncooperative with infinite time horizon. Players are assumed to be risk-neutral, i.e., utility is a linear function of the share received. Information is both complete and perfect, and everything is common knowledge.

The two subsections state the relevant parameters and the bargaining procedure, respectively.

2.1. The communication structure

Let $N = \{1, \ldots, n\}$ be the non empty but finite *set of players.* Among these, one player is randomly assigned the role of a proposer in each round. Denote α_i to be the *proposer probability* of player i. The corresponding vector $\alpha = (\alpha_i)_{i \in N}$ defines the *proposer distribution* over players in N. Given some proposer i, he makes an offer to coalition S with probability σ_{iS}. Let $\sigma_{iS} = 0$ for all S not containing i, i.e., players propose to coalitions containing them only.[1] For each player i, denote $\sigma_i = (\sigma_{iS})_{S \subset N}$ as his *coalition distribution.* We call $\sigma = (\sigma_i)_{i \in N}$ the profile of coalition distributions. In addition, players are assumed to have time preferences represented by the vector of *discount factors* $\delta = (\delta_i)_{i \in N}$. Let $0 \leq \delta_i < 1$ for each of the possibly different discount factors. According to these, players individually discount their future payoffs at the end of any round. With this, a *randomly matched bargaining game* is given by the tuple $(N, \alpha, \sigma, \delta)$.[2]

Given a profile of coalition distributions σ, the probability that player j is in a coalition responding to player i is given by $\beta_{ij} = \sum_{S \ni i,j} \sigma_{iS}$.[3] We refer to $\beta_i = (\beta_{ij})_{j \in N}$ as i's vector of respondent probabilities and to $\beta = (\beta_i)_{i \in N}$ as a profile of respondent probabilities. Note that any such profile can be induced by at least one profile of coalition distributions.[4]

[1] Technically, this makes it possible that some player is able to pursue the activity on his own.

[2] Note that $(N, (\alpha_i \sigma_{iS})_{i \in S \subset N})$ constitutes a structure equivalent to a directed and weighted hypergraph. Thus, communication structures considered in this paper are much more general than those in similar work.

[3] Thus, by definition of coalition probabilities, it is $\beta_{ii} = 1$ for all i.

[4] Given some profile of respondent probabilities β, consider the coalition probabilities σ with $\sigma_{iS} = \prod_{j \in S} \beta_{ij} \cdot \prod_{j \notin S} (1 - \beta_{ij})$. This profile of coalition distributions corresponds with independent responses from other players and induces the original respondent probabilities β.

2.2. The bargaining procedure

Given a randomly matched bargaining game $(N, \alpha, \sigma, \delta)$, any round takes the following course:

(i) A player i is randomly assigned the role of a proposer according to the proposer distribution α.

(ii) Conditional on i being proposer, a coalition $S \ni i$ is randomly chosen according to i's coalition distribution σ_i.

(iii) Proposer i suggests a division of the surplus of 1 between the members of S.

(iv) The respondents $j \in S \backslash \{i\}$ decide in an arbitrary order - which one is irrelevant - on the proposal's acceptance or rejection.

(v) If at least one respondent rejects the proposer's offer, the game proceeds to the next round and all future payoffs are individually discounted according to δ.

However, if all respondents agree to the suggested division, the surplus is divided accordingly. Players outside of coalition S receive nothing and the game ends.

3. Stationary subgame perfect equilibria

In this section, we derive and investigate stationary subgame perfect equilibria (SSPE). That is, players' offers and decisions about acceptance or rejection are homogeneous in time and independent of previous rounds.

The first subsection contains a characterization of SSPE by the corresponding payoffs, the second the statement of existence and uniqueness.

3.1. The rationality characterization

The stationary strategy of each player i is a pair (o_i, r_i). The first component is a collection of offers $o_i = (o_{iS})_{S \ni i}$. Each $o_{iS} = (o^j_{iS})_{j \in S}$ constitutes an allocation of the surplus of 1 proposed to coalition $S \ni i$. The second component r_i is a decision function stating whether to accept or reject each possible offer.

Due to stationarity, the vector of payoffs $\mu = (\mu_i)_{i \in N}$ is the same at the beginning of every round. On the condition that all offers are accepted, we find

$$\mu_i = \sum_j \alpha_j \sum_{S \ni i,j} \sigma_{jS} o^i_{jS}, \qquad i = 1, \dots, n. \quad (3.1)$$

From this we see that, given that all offers are individually rational (i.e., the shares are nonnegative) or feasible (i.e., the sum of shares does not exceed 1), the respective property also holds for the corresponding payoffs.

3.1 Proposition *Let $((o_i, r_i))_{i \in N}$ be a profile of stationary strategies with the corresponding individually rational and feasible vector of payoffs μ. The profile constitutes a SSPE iff it holds that*[5]

i) for all i and $S \ni i$, offers o_{iS} are given by

$$\begin{aligned} o^j_{iS} &= \delta_j \mu_j, && \forall j \in S \backslash \{i\}, \text{ and} \\ o^i_{iS} &= 1 - \sum_{j \in S \backslash \{i\}} \delta_j \mu_j. && \end{aligned} \quad (3.2)$$

So the shares respondents are offered are independent of both the proposer's identity and the coalition.

ii) for all i, the decision function r_i is given by

$$r_i(o_{jS}) = \begin{cases} \text{'yes'} & \text{if } o^i_{jS} \geq \delta_i \mu_i, \\ \text{'no'} & \text{otherwise.} \end{cases} \qquad \forall j, S \ni i, j. \quad (3.3)$$

In SSPE, the decision about a proposal's acceptance solely depends on the share being offered. Respondents accept shares if and only if they are not exceeded by their reservation value.

It follows that a SSPE is fully characterized by its vector of payoffs μ.

Proof Firstly, consider a respondent i. In case that an offer is rejected, he can expect $\delta_i \mu_i$ by continuation of the game in the next round. So, if the offered share exceeds his reservation value, voting 'yes' is a best response (and this is unique if all other respondents vote 'yes'). Being offered his reservation value $\delta_i \mu_i$, the respondent is indifferent and can well vote 'yes' (in fact, acceptance in this case is necessary in order to find a best response of the proposer). Offered less than $\delta_i \mu_i$, voting 'no' is a best response (unique if all other respondents vote 'yes').

[5] One can see from the proof that the decision functions r_i are uniquely determined only when all other respondents vote 'yes'. However, any changes in the cases where someone votes 'no' do not alter the outcome.

Now, consider a proposer i. Given the behavior of respondents in S, an offer corresponding with (3.2) is the most profitable among all accepted offers. Moreover, due to $1 - \sum_{j \in S \setminus \{i\}} \delta_j \mu_j > \delta_i \mu_i$, making this offer is strictly better than continuation of the game in the next round. Thus, an offer of this kind is the unique best response. □

3.2. Existence and uniqueness

By the previous proposition we know that immediate agreement is reached in case of all matches. Moreover, players' offers are known. Thus, we can rearrange equation (3.1) to find

3.2 Proposition *An individually rational and feasible vector μ is supported by a SSPE iff it solves the* bargaining equation system (BES)

$$(1 - \sum_{j \neq i} \alpha_j \sum_{S \ni i,j} \sigma_{jS}\, \delta_i)\, \mu_i + \sum_{j \neq i} \alpha_i \sum_{S \ni i,j} \sigma_{iS}\, \delta_j\, \mu_j \;=\; \alpha_i, \qquad i = 1, \ldots, n. \quad (3.4)$$

Using the definition of respondent probabilities, the BES is equivalent to

$$(1 - \sum_{j \neq i} \alpha_j\, \beta_{ji}\, \delta_i)\, \mu_i + \sum_{j \neq i} \alpha_i\, \beta_{ij}\, \delta_j\, \mu_j \;=\; \alpha_i, \qquad i = 1, \ldots, n. \quad (3.5)$$

In particular, the actual coalition structure is irrelevant insofar as the correlation of respondents does not influence the players' payoffs..

Proof Assume μ is supported by a SSPE and players' strategies are given as in proposition 3.1. In this case, equation systems (3.1) and (3.4) are equivalent. Thus, we find the BES valid.

Conversely, assume μ satisfies (3.4). Consider strategies as in proposition 3.1. Then again, equivalence of (3.1) and (3.4) shows that the payoffs from these strategies are given by μ. Thus, by proposition 3.1, these strategies constitute a SSPE. □

Thus, finding SSPE of a randomly matched bargaining game amounts to solving the corresponding BES. Still, there is the question whether each BES provides a solution, how many solutions exist, and whether solutions are both individually rational and feasible. Lemmata A.1 and A.2 in the appendix provide the necessary statements. We conclude with

3.3 Theorem (Existence and Uniqueness) *For any randomly matched bargaining game $(N, \alpha, \sigma, \delta)$, there is one unique SPE in stationary strategies. This is characterized by the vector of payoffs $\mu = (\mu_i)_{i \in N}$, i.e., the unique and always existing solution of the corresponding BES. In this SSPE, an agreement is reached in the first round.*

Proof Existence and uniqueness of solutions μ to any BES follows from lemma A.1 in the appendix. Lemma A.2 states this solution μ constitutes an individually rational and feasible vector of payoffs. Thus, the desired statement follows from proposition 3.2. □

4. Properties of SSPE payoffs

We now present a variety of basic properties as well as monotonicties of the SSPE payoffs. For some (at first) intuitive monotonicities, we find they do not show in all cases. Examples are provided for illustration.

4.1. Basic properties

Before listing the basic properties, we introduce some further definitions.

For any permutation π on N, define the *permuted game* $\pi(G) = (N, \pi(\alpha), \pi(\sigma), \pi(\delta))$ by

$$\pi(\alpha) = (\alpha_{\pi^{-1}(i)})_{i \in N},$$
$$\pi(\sigma) = (\sigma_{\pi^{-1}(i)\pi^{-1}(S)})_{i \in N, S \subset N}, \text{ and}$$
$$\pi(\delta) = (\delta_{\pi^{-1}(i)})_{i \in N}.$$

Of course it also holds that $\pi(\beta) = (\beta_{\pi^{-1}(i)\pi^{-1}(j)})_{i,j \in N}$ is the matrix of respondent probabilities of the permuted social network.

The game is called *symmetric* with respect to π if $G = \pi(G)$. For a symmetric game, it holds $\pi(\beta) = \beta$.

A player i is called a *null player* if $\alpha_i = 0$.

He is called a *null receiver* if $\delta_i = 0$ or $\alpha_i = 0$.

4.1 Proposition

(i) Smoothness: *The SSPE payoffs μ are continuous and infinitely often continuously differentiable in all parameters.*

(ii) Individual rationality and efficiency: *The SSPE payoffs μ are individually rational and efficient, that is, it holds*

$$\mu \geq 0 \tag{4.1}$$

and

$$\sum_i \mu_i = 1. \tag{4.2}$$

(iii) Anonymity: *Let π be a permutation on N. Then $\pi(\mu)$ is the vector of payoffs in the permuted game $\pi(G) = (N, \pi(\alpha), \pi(\sigma), \pi(\delta))$.*

(iv) Symmetry: *Let π be a permutation on N such that game G is symmetric with respect to π. Then it is $\pi(\mu) = \mu$.*

(v) Null players: *For all i, it is*

$$\mu_i = 0 \Leftrightarrow \alpha_i = 0, \tag{4.3}$$

meaning that a player expects nothing from bargaining if and only if he is a null player.

So it also holds that a player receives the whole bargaining value if and only if all other players are null players. For any i, it is

$$\mu_i = 1 \Leftrightarrow \alpha_i = 1. \tag{4.4}$$

(vi) Null receivers: *If a player i is a null receiver, he has a reservation value of 0. It then holds that*

$$\mu_i \leq \alpha_i. \tag{4.5}$$

If all of a player i's possible respondents are null receivers, it is

$$\mu_i \geq \alpha_i. \tag{4.6}$$

If all possible respondents are null receivers, it is

$$\mu = \alpha. \tag{4.7}$$

Proof

(i) See lemma A.1 in the appendix.

(ii) See lemma A.2 in the appendix.

(iii) The BES of the permuted game is just the original one with permuted rows and columns. So $\pi(\mu)$ constitutes the unique solution.

(iv) If this symmetry holds, the BES of the permuted game is identical with the original one. Hence, the solution to the original BES also solves the permuted BES. Since solutions are unique, μ and $\pi(\mu)$ coincide.

(v) For $\alpha_i = 0$, row i of the BES becomes $(1 - \sum_{j \neq i} \alpha_j \beta_{ji} \delta_i)\mu_i = 0$. This holds iff $\mu_i = 0$. Thus, $\mu_i = 1 \Leftrightarrow \alpha_i = 1$ follows directly from (ii).

(vi) That null receivers have a reservation value of 0 follows from (v). The remaining statements follow from the BES and (ii). □

4.2. Monotonicities in proposer probabilities

Note that, for all i and $S \ni i$, it holds

$$0 \leq \delta_i \mu_i < 1 - \sum_{j \in S \setminus \{i\}} \delta_j \mu_j, \tag{4.8}$$

and hence,

(i) it is never bad to be involved in bargaining.

(ii) when being involved in bargaining, it is always better to be proposer than being respondent.

Thus, one could expect a player's payoff to rise when proposer probability is shifted from a third player to him. At least, this shift always increases the chance to be proposer and never decreases the chance to be involved in bargaining.

Nonetheless, it *may* hold for the directional derivative corresponding with a shift of proposer probability from one player j to another player i that

$$\frac{\partial \mu_i}{\partial \alpha_i} - \frac{\partial \mu_i}{\partial \alpha_j} < 0. \tag{4.9}$$

This is shown in the following example:

4.2 Example *Consider the communication structure as depicted in figure 1 with parameters* $\alpha = (0.6, \alpha_2, \alpha_3)$, $\sigma_{1\{1,2\}} = 1$, $\sigma_{2\{1,2\}} = 1$, $\sigma_{3\{2,3\}} = 1$, *and* $\delta = (0.95, 0.95, 0.95)$. [6]

The corresponding payoffs μ *as a function of* α_2 *are depicted in figure 2.*

[6] In the graph from figure 1, an edge from i to j symbolizes the possibility that i proposes to the coalition containing him *and* j, that is, coalition $\{i, j\}$. In the author's opinion, graphs are better to read than hypergraphs.

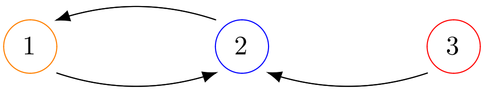

Figure 1: Three players on a line with a restricted central player

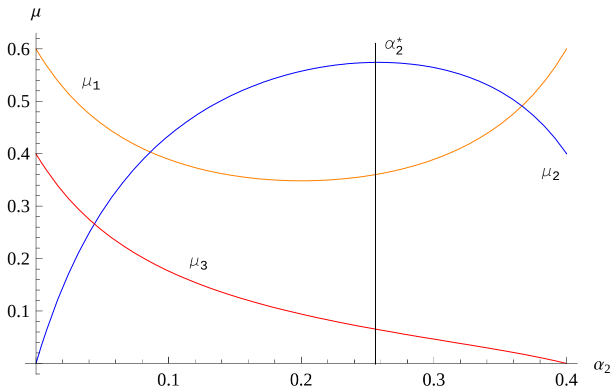

Figure 2: Payoffs μ as a function of α_2 (where $\alpha_3 = 0.4 - \alpha_2$)

Note that, for $\alpha_2 > \alpha_2^*$, it is $\frac{\partial \mu_2}{\partial \alpha_2} - \frac{\partial \mu_2}{\partial \alpha_3} < 0$, i.e., shifts of proposer probability from player 3 to player 2 reduce the latter's payoff. This comes from the fact that, as $\alpha_2 \to 0.4$, the system becomes effectively a two-player system, and player 2 loses his structural advantage. In particular, player 1 profits from being respondent more often which in turn decreases 2's share as a proposer.

However, we find the weaker, more specific monotonicity:

4.3 Proposition *For all players i, it holds for the directional derivative corresponding with a shift of the proposer distribution α in direction of the one-point distribution e_i that*[7]

$$(1 - \alpha_i)\frac{\partial \mu_i}{\partial \alpha_i} - \sum_{j \neq i} \alpha_j \frac{\partial \mu_i}{\partial \alpha_j} \geq 0. \tag{4.10}$$

Strict inequality holds iff $\alpha_i < 1$.

This means that at $\alpha_i = 1$, μ_i takes the unique local maximum (which is also global due to $\mu_i = 1$, cf. proposition 4.1).

[7]By e_i we denote the i-th standard vector, that is, a vector with the i'th component being 1 and all others being 0. Interpreted as a distribution, this is the one-point distribution on i.

Proof The respective directional derivative of the BES $A\mu = \alpha$ amounts to

$$
\begin{aligned}
&A\left((1-\alpha_i)\frac{\partial\mu}{\partial\alpha_i} - \sum_{j\neq i}\alpha_j\frac{\partial\mu}{\partial\alpha_j}\right)\\
&=\left((1-\alpha_i)e_i - \sum_{j\neq i}\alpha_j e_j\right) - \left((1-\alpha_i)\frac{\partial A}{\partial\alpha_i} - \sum_{j\neq i}\alpha_j\frac{\partial A}{\partial\alpha_j}\right)\mu.
\end{aligned}
$$

Because payoffs sum up to 1, it holds for the sum of overall changes that $\sum_i\left((1-\alpha_i)\frac{\partial\mu_i}{\partial\alpha_i} - \sum_{j\neq i}\alpha_j\frac{\partial\mu_i}{\partial\alpha_j}\right) = 0$. Thus, due to A's columns adding up to 1, we know that the components of the right side add up to 0 as well.

Moreover, for $j \neq i$, the j-th component of the right side amounts to

$$
\begin{aligned}
&-\alpha_j - \left(-(1-\alpha_i)\beta_{ij}\delta_j\mu_j + \sum_{k\neq i,j}\alpha_k\beta_{kj}\delta_j\mu_j - \sum_{k\neq j}\alpha_j\beta_{jk}\delta_k\mu_k\right)\\
&= -\alpha_j + \left((\beta_{ij} - \sum_{k\neq j}\alpha_k\beta_{kj})\delta_j\mu_j + \sum_{k\neq j}\alpha_j\beta_{jk}\delta_k\mu_k\right)\\
&\leq -\alpha_j + \left((1 - \sum_{k\neq j}\alpha_k\beta_{kj}\delta_j)\mu_j + \sum_{k\neq j}\alpha_j\beta_{jk}\delta_k\mu_k\right)\\
&= 0.
\end{aligned}
$$

Due to $(\beta_{ij} - \sum_{k\neq j}\alpha_k\beta_{kj})\delta_j < 1 - \sum_{k\neq j}\alpha_k\beta_{kj}\delta_j$, the inequality is strict iff $\mu_j > 0$. Hence, by the null player property, the inequality is strict for at least one $j \neq i$ iff $\alpha_i < 1$.

Thus, the desired statement follows with help of corollary A.4 in the appendix. □

While this monotonicity is of course less general than that for general shifts, changes of this kind are the most natural changes for proposer distributions.[8]

[8]Consider that each player i's offers constitute a λ_i-Poisson process, that is, they 'occur evenly' with mean frequency $\lambda_i \geq 0$ (for information on Poisson processes, see Karlin and Taylor, 1975, pp. 22-26 and 123-128). Then player i's probability to be the proposer, given that some player in fact makes an offer, is just $\alpha_i = \frac{\lambda_i}{\sum_k \lambda_k}$ (cf. Grimmet and Stirzaker, 1992, p. 243). Thus,

4.3. Monotonicities in coalition and respondent probabilities

Consider a player i to be no null player and S, T two distinct coalitions containing i. Assume $1 - \sum_{j \in S} \delta_j \mu_j > 1 - \sum_{j \in T} \delta_j \mu_j$, that is, the *excess* of coalition S is greater than that of coalition T. Although it is better for player i to propose to the weaker coalition S, it *may* hold for the directional derivative corresponding with a shift of coalition probability from T to S that

$$\frac{\partial \mu_i}{\partial \sigma_{iS}} - \frac{\partial \mu_i}{\partial \sigma_{iT}} < 0. \tag{4.11}$$

This is shown in the following example:

4.4 Example *Consider the communication structure from figure 3 with parameters* $\alpha = (0.27, 0.35, 0.38)$*,* $\sigma_{1\{1,2\}} = 1$*,* $\sigma_2 = (\sigma_{2\{1,2\}}, \sigma_{2\{2,3\}})$*,* $\sigma_{3\{2,3\}} = 1$*, and* $\delta = (0.9, 0.5, 0.9)$*.*

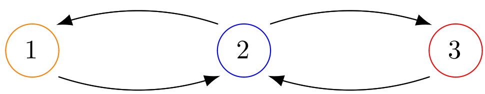

Figure 3: Three players on a line

The corresponding payoffs μ *as a function of* $\sigma_{2\{1,2\}}$ *are depicted in figure 4.*

Note that, for $\sigma_{2\{1,2\}} > \sigma^*_{2\{1,2\}}$*, it is* $\frac{\partial \mu_2}{\partial \sigma_{2\{1,2\}}} - \frac{\partial \mu_2}{\partial \sigma_{2\{2,3\}}} < 0$*, while* $\delta_1 \mu_1 < \delta_3 \mu_3$*, i.e., coalition* $\{1, 2\}$ *has a greater excess than coalition* $\{2, 3\}$*. This effect occurs because the benefit from the actual difference in reservation values* $\delta_3 \mu_3 - \delta_1 \mu_1$ *is exceeded by the weighted changes of reservation values. Player* 1*'s reservation value rises too fast and is weighted too high.*[9]

In the case of respondent probabilities, however, one finds two more general monotonicities.

---

the proposer distribution varies with an increase of i's offering frequency λ_i as follows: it is $\frac{\partial \alpha_i}{\partial \lambda_i} = \frac{1 - \alpha_i}{\sum_k \lambda_k}$ and $\frac{\partial \alpha_j}{\partial \lambda_i} = \frac{-\alpha_j}{\sum_k \lambda_k}$ for $j \neq i$. This in turn constitutes a shift of α in direction of the one-point distribution e_i.

[9] Using the short from $\partial \mu_i$ for $\frac{\partial \mu_i}{\partial \sigma_{2\{1,2\}}} - \frac{\partial \mu_i}{\partial \sigma_{2\{2,3\}}}$, we find

$$\begin{aligned}&(1 - (\alpha_1 + \alpha_3)\delta_2)\partial \mu_2 \\ &\qquad = \alpha_2(\delta_3 \mu_3 - \delta_1 \mu_1) - \alpha_2(\sigma_{2\{1,2\}} \delta_1 \partial \mu_1 + \alpha_2 \sigma_{2\{2,3\}} \delta_3 \partial \mu_3.\end{aligned}$$

From this, one sees that the weighted changes in reservation values may outweigh the actual difference in reservation values.

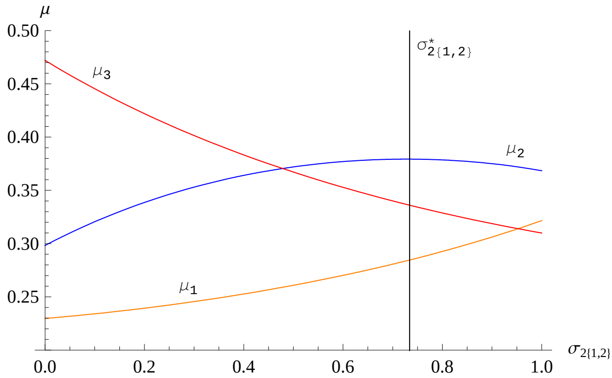

Figure 4: Payoffs μ as a function of $\sigma_{2\{1,2\}}$ (where $\sigma_{2\{1,2\}}+\sigma_{2\{2,3\}} = 1$)

4.5 Proposition *Let i, j be two distinct players where i is no null player. Then it holds for the partial derivative of the respondent probability β_{ij} that*

$$\frac{\partial \mu_i}{\partial \beta_{ij}} \leq 0, \text{ and} \tag{4.12}$$

$$\frac{\partial \mu_j}{\partial \beta_{ij}} \geq 0. \tag{4.13}$$

Strict inequalities hold iff j is no null receiver.

Proof The respective directional derivative of the BES $A\mu = \alpha$ amounts to

$$A\frac{\partial \mu}{\partial \beta_{ij}} = -\frac{\partial A}{\partial \beta_{ij}}\mu.$$

The i-th component of the right side amounts to $-\alpha_i \delta_j \mu_j$, the j-th component to $\alpha_i \delta_j \mu_j$, and all other components are 0. Thus, the desired statement follows with corollary A.4 in the appendix. □

The proposition states that, if j is no null receiver, a decrease in β_{ij} is always good for player i but bad for player j. The change is irrelevant for both players if j is a null receiver. Consequently, players prefer minimum respondent probability on third players but maximum respondent probability on themselves.

A direct consequence is that, in a special case, one finds the previously considered monotonicity for coalition probabilities to hold.

4.6 Proposition *Let i be no null player and $S, T \ni i$ two coalitions where $S \backslash T$ is either empty or contains null receivers only. Then it holds for the directional derivative corresponding with a shift of coalition probability from T to S that*

$$\frac{\partial \mu_i}{\partial \sigma_{iS}} - \frac{\partial \mu_i}{\partial \sigma_{iT}} \geq 0. \tag{4.14}$$

Strict inequality holds iff $T \backslash S$ contains at least one player who is no null receiver, that is, iff S has greater excess than T.

Proof The statement follows from proposition 4.5. □

Thus, players prefer to have coalition probability on minimal coalitions only. Still, nothing can be said about finding the optimal coalition distribution over minimal coalitions, even if their excess is different. In particular, this means that equilibria in games where coalition distributions are chosen *before* the actual bargaining are different from those in games where coalition distributions may be chosen in every round.

4.4. Monotonicities in discount factors

The second general monotonicity is found for the discount factors. An increase in a player's discount factor is never bad, and it is beneficial if there is positive probability on him being a respondent.

4.7 Proposition *Let i be no null player. Then it holds for the partial derivative of his discount factor δ_i that*

$$\frac{\partial \mu_i}{\partial \delta_i} \geq 0. \tag{4.15}$$

Strict inequality holds iff i is a possible respondent.

Proof The respective directional derivative of the BES $A\mu = \alpha$ amounts to

$$A \frac{\partial \mu}{\partial \delta_i} = -\frac{\partial A}{\partial \delta_i} \mu.$$

The i-th component of the right side amounts to

$$\sum_{j \neq i} \alpha_j \beta_{ji} \mu_i,$$

any j-th component to

$$-\alpha_j \beta_{ji} \mu_i.$$

Thus, the desired statement follows with corollary A.4 in the appendix. □

5. Bargaining power

As players have positive time preferences, a friction is placed upon them which is one reason for existence and uniqueness of stationary subgame perfect equilibria. Considering arbitrarily patient players brings - apart from the still exogenous communication structure - a cooperational aspect into the game.

The following subsection states the definition of power by considering the limit of SSPE payoffs and shortly lists the induced properties. The second subsection discusses the newly found veto player property.

5.1. The transition to infinite patience

In cooperational games, players' bargaining power is usually understood as what they can secure when facing the resistance of everybody else. So in this game, where minimizing other players' share amounts to maximizing the own, bargaining power can be considered to be the payoff of infinite patient players.

We thus come to the following definition. Given some communication structure (N, α, σ), denote the players' bargaining power Φ by

$$\Phi = \lim_{\delta \to 1} \mu(\delta).$$

Here, $\mu(\delta)$ is the vector of SSPE payoffs in the corresponding randomly matched bargaining games where players have a common discount factor δ. Thus, bargaining power is defined to be the limit of payoffs as the common discount factor converges to 1.[10] From lemmata A.1 and A.2 in the appendix, we know that $\mu(\delta)$ is a bounded rational function in δ. Hence, the limit Φ is well defined.

The properties we found valid for SSPE payoffs in the case of imperfectly patient players may or may not hold for bargaining power. Due to/in spite of taking the limit, they now are

invalid: Smoothness (proposition 4.1 (i)) does not hold. Bargaining power may even be discontinuous (cf. the example in the next section).

valid: Individual rationality and efficiency (proposition 4.1 (ii)) as well as anonymity and symmetry (proposition 4.1 (iii) and (iv)) are not affected by considering the limit of payoffs.

[10] However, it is also possible to consider bargaining power in case of different time preferences. With individual interest rates $r_i > 0$ and time $\Delta > 0$ between two successive rounds, discount factors amount to $\delta_i = e^{-r_i \Delta}$. Bargaining power then equals the limit of payoffs as $\Delta \to 0$.

weakened: The null player property (proposition 4.1 (v)) now is an implication only ($\alpha_i = 0 \Rightarrow \Phi_i = 0$). The monotonicities in proposer, respondent, and coalition probabilities (propositions 4.3, 4.5, and 4.6) become weak monotonicities in any case.

5.2. Veto players and discontinuities of power

However, due to players' infinite patience, one additional property can be found. We call a player i a veto player if he is a member of all possible coalitions, i.e., if $\beta_{ji} = 1$ for all j with $\alpha_j > 0$.

5.1 Proposition *Let $V \neq \emptyset$ be the set of veto players which are no null players.*

Then, if there is no player $j \notin V$ for which

i) $\alpha_j > 0$ and

ii) $\forall i \in V$: $\beta_{ij} = 1$,

it holds

$$\sum_{i \in V} \Phi_i = 1.$$

Proof Assume $\Phi_j > 0$ for some $j \notin V$. Then there is $i \in V$ with $\beta_{ij} < 1$ and for this it holds

$$(1 - \sum_{k \neq i} \alpha_k)\Phi_i + \sum_{k \neq i} \alpha_i \beta_{ik} \Phi_k = \alpha_i(\Phi_i + \sum_{k \neq i} \beta_{ik} \Phi_k) < \alpha_i.$$

This contradicts the limit of row i of the BES. □

The proposition states that if there are veto players who are no null players, and no other player who is no null player can block all their offers, then they hold all bargaining power.

We present an example.

5.2 Example *Consider the communication structure depicted figure 1 with parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\sigma_{1\{1,2\}} = 1$, $\sigma_{2\{1,2\}} = 1$, and $\sigma_{3\{2,3\}} = 1$.*

For this, the players' bargaining power is given as follows:

$$\begin{aligned} \alpha_2 = 0: &\quad \Phi = (\alpha_1, 0, \alpha_3) \\ \alpha_3 = 0: &\quad \Phi = (\alpha_1, \alpha_2, 0) \\ \alpha_2, \alpha_3 > 0: &\quad \Phi = (0, 1, 0) \end{aligned}$$

Here we see that arbitrarily small but positive proposer probability may be sufficient to hold all power. This and the null player property show up a discontinuity of bargaining power.

In addition, note that a shift of proposer probability from 2 to 3 may be beneficial also in the case of infinitely patient players. In fact, it is necessary for player 2's holding all power that player 3 is able to make offers – otherwise, player 1 would also be a veto player. Interestingly, this is analogous to the donation paradox of power indices, that is, a shift of voting weight from one player to another reduces the latter's payoff (see Felsenthal and Machover, 1991, p. 252). Besides the null player property, this is another analogy of proposer probabilities and voting weights.[11]

6. Concluding remarks

We have derived existence and uniqueness of stationary subgame perfect equilibria (SSPE) for all randomly matched bargaining games. The main emphasis of the work has been on matching dynamics that do not assume symmetry as for instance those of Berg (1999) or Calvó-Armengol (2001). A great variety of properties has been shown, most notably smoothness in all parameters, efficiency, null players, and monotonicities for each group of parameters. Some supposedly intuitive monotonicties do, however, not show in all cases. This fact highlights a dichotomy of preferences over probability distributions on the one hand and preferences over outcomes of respective randomizations on the other.

Payoffs have been found to be interpretable as bargaining power when players grow infinitely patient. In this rather cooperational setting, veto players (most often) hold all power. Rather inconvenient are yet the possible discontinuities of bargaining power. Since so far it seems that discontinuities only occur at the borders of the parameter space, the problem might be overcome by a refinement similar to trembling-hand perfectness.[12] [13]

---

[11] I thank J.M. Alonso-Meijide for pointing out this analogy.

[12] Discontinuities are expected only where small changes in parameters are critical to the validity of the veto player property, see proposition 5.1.

[13] Consider the ϵ-perturbed version of randomly matched bargaining game $(N, \alpha, \sigma, \delta)$ as the corresponding game where each player has a proposer probability of at least ϵ. Let Φ^ϵ be the vector of bargaining power in the perturbed game and consider $\Phi = \lim_{\epsilon \to 0}$ as the refined, 'trembling-hand perfect' vector of bargaining power of game $(N, \alpha, \sigma, \delta)$. This makes sense insofar as the extreme cases where some players are not make any offer are ruled out. Also, incomplete information about whether each player has (infinitesimally small) positive proposer probability or none should produce the same result.

Irrespective of how perturbations are designed, the discontinuities in example

Based on the established results, it is possible to measure power in committees by use of randomly matched bargaining games. Given a simple game, matching dynamics can be induced which then yield the players' (bargaining) power. Note, however, that veto players' holding all power is a necessary property of core allocation which is not satisfied by any 'traditional' power index (cf. Shapley, 1953; Banzhaf, 1965; Deegan and Packel, 1978; Holler and Packel, 1983). This in turn implies that no such index can be induced by a proper communication structure. Instead, using for instance matching dynamics which put equal probability on each minimal winning coalition and equal probability on each member to be the proposer would yield a power index rather similar to the nucleolus (cf. Montero, 2006).

5.2 are overcome.

A. Mathematical appendix

In the appendix, important properties of (3.5), the bargaining equation system (BES), are presented.

For this, note first that the BES corresponds with

$$A\,\mu \;=\; \alpha, \tag{A.1}$$

where $A = (A_{ij})_{ij\in N}$ is a matrix with

$$a_{ii} = 1 - \sum_{j\neq i} \alpha_j\,\beta_{ji}\,\delta_i \qquad \forall i, \text{ and} \tag{A.2}$$

$$a_{ij} = \alpha_i\,\beta_{ij}\,\delta_j \qquad \forall i \neq j. \tag{A.3}$$

We call any matrix A a *bargaining matrix* if it corresponds with the BES of some social network and a vector of discount factors.

The first lemma deals with existence and uniqueness of solutions to the BES.

A.1 Lemma *Any bargaining matrix A is regular. Hence, there is always a unique solution $\mu = A^{-1}\alpha$ to the BES.*

Every component of this solution is a rational function in all components of α, β, and δ. Hence, the solution is continuous and infinitely often continuously differentiable in all parameters.

Proof Assume there exists $x \neq 0$ with $A\,x \;=\; 0$. We show a contradiction:

Since A is nonnegative and the diagonal is positive, the sets $N_+ = \{i \in N \mid x_i > 0\}$ and $N_- = \{i \in N \mid x_i < 0\}$ are both nonempty. Note that $x_i \neq 0$ implies $\alpha_i > 0$.

With $1 - \sum_{j\in N_-} \alpha_j\beta_{ji}\delta_i \geq \sum_{j\in N_+} \alpha_j$, we find

$$\begin{aligned}\sum_{i\in N} \delta_i x_i &< \sum_{i\in N_+} \frac{1-\sum_{j\in N_-} \alpha_j\beta_{ji}\delta_i}{\sum_{j\in N_+} \alpha_j} x_i + \sum_{i\in N_-} \frac{\sum_{j\in N_+} \alpha_j\beta_{ji}}{\sum_{j\in N_+} \alpha_j} \delta_i x_i \\ &= 0.\end{aligned}$$

Analogously, it is

$$\begin{aligned}\sum_{i\in N} \delta_i x_i &> \sum_{i\in N_-} \frac{1-\sum_{j\in N_+} \alpha_j\beta_{ji}\delta_i}{\sum_{j\in N_-} \alpha_j} x_i + \sum_{i\in N_+} \frac{\sum_{j\in N_-} \alpha_j\beta_{ji}}{\sum_{j\in N_-} \alpha_j} \delta_i x_i \\ &= 0,\end{aligned}$$

a contradiction to the previous inequality. Hence, there is no $x \neq 0$ with $A\,x \;=\; 0$. □

The second ensures the individual rationality and feasibility of any solution to a BES.

A.2 Lemma *For any solution x of a BES, it holds*

$$x \geq 0 \tag{A.4}$$

and

$$\sum_i x_i = 1. \tag{A.5}$$

Proof

($x \geq 0$) Let $N_+ = \{i \in N \mid x_i > 0\}$ and $N_- = \{i \in N \mid x_i < 0\}$ and note that $x_i \neq 0$ implies $\alpha_i > 0$. Since A is nonnegative, N_+ must be nonempty. We show $N_- = \emptyset$ by a contradiction:

With $1 - \sum_{j \in N_-} \alpha_j \beta_{ji} \delta_i \geq \sum_{j \in N_+} \alpha_j$, we find

$$\begin{aligned}\sum_{i \in N} \delta_i x_i &< \sum_{i \in N_+} \frac{1 - \sum_{j \in N_-} \alpha_j \beta_{ji} \delta_i}{\sum_{j \in N_+} \alpha_j} x_i + \sum_{i \in N_-} \frac{\sum_{j \in N_+} \alpha_j \beta_{ji}}{\sum_{j \in N_+} \alpha_j} \delta_i x_i \\ &= 1.\end{aligned}$$

Now assume that $N_- \neq \emptyset$. As above, we find

$$\begin{aligned}\sum_{i \in N} \delta_i x_i &> \sum_{i \in N_-} \frac{1 - \sum_{j \in N_+} \alpha_j \beta_{ji} \delta_i}{\sum_{j \in N_-} \alpha_j} x_i + \sum_{i \in N_+} \frac{\sum_{j \in N_-} \alpha_j \beta_{ji}}{\sum_{j \in N_-} \alpha_j} \delta_i x_i \\ &= 1,\end{aligned}$$

a contradiction to the previous inequality. Hence, it must be $N_- = \emptyset$.

($\sum_i x_i = 1$) The sum of elements in each column of the bargaining matrix is 1. By this, it is

$$\sum_i x_i = \sum_i (\sum_j a_{ji}) x_i = \sum_j (\sum_i a_{ji} x_i) = \sum_j \alpha_j = 1. \qquad \square$$

The third and final lemma shows an important property of the inverse of a bargaining matrix. This is necessary to proof the monotonicities in section 4.

A.3 Lemma *Let $A^{-1} = (a_{ij}^{-1})_{i,j \in N}$ denote the inverse of a bargaining matrix A. Then for any i and any $j \neq i$ it is*

$$a_{ii}^{-1} > a_{ij}^{-1}, \tag{A.6}$$

meaning that in all of A^{-1}'s rows the diagonal element is the strictly greatest element.

Proof Consider the LES $Ax = e_i - e_j$ with $i \neq j$.

Firstly, we show a contradiction from the assumption $x_i = 0$. In this case, the sum of rows i and j amounts to

$$(1 - \sum_{l \neq i,j} \alpha_l \beta_{lj} \delta_j) x_j + \sum_{l \neq i,j} (\alpha_i \beta_{il} + \alpha_j \beta_{jl}) \delta_l x_l = 0,$$

rows $k \neq i, j$ are

$$(1 - \sum_{l \neq k} \alpha_l \beta_{lk} \delta_k) x_k + \sum_{l \neq k} \alpha_k \beta_{kl} \delta_l x_l = 0.$$

The matrix corresponding with this LES is a bargaining matrix. So it must be $x_l = 0$ for $l \neq i$ and also $x = 0$, a contradiction to $Ax = e_i - e_j$. Hence it cannot be $x_i = 0$.

Since x_i is continuous in all parameters and the set of all possible parameters is connected, it is either always $x_i < 0$ or always $x_i > 0$. For $\alpha_i = 0$, row i states $(1 - \sum_{l \neq i} \alpha_l \beta_{li} \delta_i) x_i = 1$. So it is $x_i > 0$ in this case and hence for all α, β, and δ.

The actual statement follows from the fact that $x = A^{-1}(e_i - e_j)$ is the difference of the i-th and j-th column of A^{-1}. □

A.4 Corollary *For a bargaining matrix A, consider the situation*

$$Ax = r, \tag{A.7}$$

where $r_i > 0$ for some i, $r_j \leq 0$ for $j \neq i$, and $\sum_k r_k = 0$. It then holds that

$$x_i > 0. \tag{A.8}$$

References

J. Banzhaf. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review*, 19: 317–343, 1965.

D.P. Baron and J.A. Ferejohn. Bargaining in legislatures. *American Political Science Review*, 83: 1181–1206, 1989.

C. Berg. Verhandlungen in Netzwerken - Graphspiele und ihre Anwendungen. In *Beiträge zur Verhandlungstheorie*, pages 109–197. PhD thesis, Universität Hamburg, 1999.

A. Calvó-Armengol. Bargaining power in communication networks. *Mathematical Social Sciences*, 41: 69–87, 2001.

J. Deegan and E.W. Packel. A new index of power for simple n-person games. *International Journal of Game Theory*, 7: 113–123, 1978.

H. Eraslan. Uniqueness of stationary equilibrium payoffs in the Baron-Ferejohn model. *Journal of Economic Theory*, 103: 11–30, 2002.

D.S. Felsenthal and M. Machover. *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes.* The MIT Press, Cambridge, 1991.

G. R. Grimmet and D. R. Stirzaker. *Probability and Random Processes.* Oxford University Press, Oxford, 1992.

M.J. Holler and E.W. Packel. Power, luck and the right index. *Journal of Economics*, 43(1): 21–29, 1983.

S. Karlin and H.M. Taylor. *A First Course in Stochastic Processes.* Academic Press, San Diego, 1975.

M. Montero. Noncooperative foundations of the nucleolus in majority games. *Games and Economic Behavior*, 54: 380–397, 2006.

R. Myerson. Graphs and cooperation in games. *Mathematics of Operation Research*, 2: 225–229, 1977.

A. Okada. A Noncooperative Coalitional Bargaining Game with Random Proposers. *Games and Economic Behavior*, 16: 97–108, 1996.

G. Owen. Values of graph-restricted games. *SIAM Journal on Algebraic Discrete Methods*, 7: 210–220, 1986.

E. Rosenthal. Communication networks and their role in cooperative games. *Social Networks*, 10: 255–263, 1988.

A. Rubinstein. Perfect Equilibrium in a Bargaining Model. *Econometrica*, 50: 97–111, 1982.

L.S. Shapley. A Value for n-Person Games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, II*, volume 28 of *Annals of Mathematical Studies*, pages 307–317. Princeton University Press, Princeton, 1953.